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ON GENERALIZATIONS OF TWO CURIOUS DIVISIBILITY PROPERTIES

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Abstract. In this paper, we extend two curious divisibility properties for the general second order linear recurrence $\{U_n(p,q)\}$. We also give new recursive identities for the general second linear recurrences $\{U_n(p,q)\}$ and $\{V_n(p,q)\}$. These results generalize the results given by E. Kılıç, "A matrix approach for generalizing two curious divisibility properties", Miskolc Math. Notes, vol. 13., No. 2, pp. 389-396, 2012.

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1. Introduction

Let p and q be nonzero integers such that $p^2+4q \neq 0$. For $n > 1$, the generalized Fibonacci sequence $\{U_n(p,q)\}$ and the generalized Lucas sequence $\{V_n(p,q)\}$ are defined by

$$U_n(p,q) = pU_{n-1}(p,q) + qU_{n-2}(p,q)$$

and

$$V_n(p,q) = pV_{n-1}(p,q) + qV_{n-2}(p,q),$$

where $U_0(p,q) = 0,\ U_1(p,q) = 1$ and $V_0(p,q) = 2,\ V_1(p,q) = p$, respectively.

Let α and β be the roots of the equation $x^2 - px - q = 0$. Then the Binet formulas of the sequences $\{U_n(p,q)\}$ and $\{V_n(p,q)\}$ are given by

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \ \text{ and } \ V_n(p,q) = \alpha^n + \beta^n$$

If $p = q = 1$, then $U_n(1,1) = F_n$ (nth Fibonacci number) and $V_n(1,1) = L_n$ (nth Lucas number).

It is a well known fact that

$$\gcd(F_n,\ F_m) = F_{\gcd(n,m)}.$$

It is also known that F_{kn} is a multiple of F_n, for all integers k and n. In [9], the author showed that, for $n > 2$, the Fibonacci number F_m is a multiply of F_n^2 if and only if

m is multiply of nF_n (for more details see [4]). Also, in [1], the author obtained the following divisibility properties:

i) $F_{kn-1} - F_{n-1}^k$ is divisible by F_n^2,

ii) $F_{kn-2} - (-1)^{k+1} F_{n-2}^k$ is divisible by F_n^2,

where $n,k \geq 1$. Recently Kılıç [7] generalized these results for a general second order linear recursion $\{U_n(p,1)\}$ as follows:

$$U_r^{k-1}(p,1)U_{kn-r}(p,1) - (-1)^{(r-1)(k+1)} U_{n-r}^k(p,1) \text{ is divisible by } U_n^2(p,1).$$

Furthermore, the author found new recursive identities for the general second order linear recurrences $\{U_n(p,1)\}$ and $\{V_n(p,1)\}$.

In this paper, for the case $q \neq 1$, we show that

$$U_r^{k-1}(p,q)U_{kn-r}(p,q) - (-1)^{(r-1)(k+1)} q^{r(k-1)} U_{n-r}^k(p,q) \text{ is divisible by } U_n^2(p,q).$$

To do that we use matrix methods. Matrix methods are useful tools for derivating some properties of linear recurrences (see [3, 5, 6, 8, 10]). We consider the quotient

$$\frac{U_r^{k-1}U_{kn-r} - (-1)^{(r-1)(k+1)} q^{r(k-1)} U_{n-r}^k}{U_n^2},$$

where $n,k \geq 1$. We define a generating matrix for this quotient for fixed n and increasing values of k. Then we give an explicit statement for the quotient. Also, by considering this explicit statement, we find new recursive identities for the general second order linear recurrences $\{U_n(p,q)\}$ and $\{V_n(p,q)\}$. Thus we obtain a generalization of the results given in [7].

Throughout this study, for simplicity, we will denote $U_n(p,q)$ by U_n and $V_n(p,q)$ by V_n.

2. Main Results

Before we give our main results, we need some auxiliary results and definitions. Denote the quotient $\left(U_r^{k-1}U_{kn-r} - (-1)^{(r-1)(k+1)} q^{r(k-1)} U_{n-r}^k\right)/U_n^2$ by $s(n,k)$. Define two matrices $H(n)$ and $G(n,k)$ of order 3 as follows:

$$H(n) = \begin{bmatrix} A_{n-1} & B_n & -(-q)^{n+r} U_r^2 U_{n-r} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$G(n,k) = \begin{bmatrix} s(n,k+2) & t(n,k+2) & -(-q)^{n+r} U_r^2 U_{n-r} s(n,k+1) \\ s(n,k+1) & t(n,k+1) & -(-q)^{n+r} U_r^2 U_{n-r} s(n,k) \\ s(n,k) & t(n,k) & -(-q)^{n+r} U_r^2 U_{n-r} s(n,k-1) \end{bmatrix},$$

where

$$A_{n-1} = U_r V_n - (-q)^r U_{n-r},$$

$$B_n = (-q)^r U_r V_n U_{n-r} - (-q)^n U_r^2$$

and

$$t(n,k) = \Big((-q)^r U_r^{k-1} U_{kn} U_{n-r}^2 + (-1)^{r-1} q^{2n-r} U_r^{k+1} U_{n(k-2)} + (-1)^{(r+1)(k-1)} q^{r(k-1)} U_r U_{2n} U_{n-r}^k\Big) / U_n^3$$

Lemma 1. *For $n \geq 1$, the eigenvalues of $H(n)$ are $U_r\alpha^n$, $U_r\beta^n$ and $-(-q)^r U_{n-r}$.*

Proof. The characteristic polynomial of $H(n)$ is

$$x^3 - A_{n-1}x^2 - B_n x + (-q)^{n+r} U_r^2 U_{n-r} = 0$$

and it is factorized as

$$(x - U_r\alpha^n)(x - U_r\beta^n)(x + (-q)^r U_{n-r}) = 0,$$

as required. □

Thus the first main result of this paper is the following.

Theorem 1. *For $n > 1$,*

$$H(n)^k = G(n,k).$$

Proof. In the proof, we will use induction on k. Since $G(n,1) = H(n)$, the result is true when $k = 1$. Now assume that $H(n)^{k-1} = G(n,k-1)$. Then, by the definitions of $s(n,k)$ and $t(n,k)$, we have

$$A_{n-1}s(n,k+1) + t(n,k+1) = s(n,k+2)$$

and

$$B_n s(n,k+1) - (-q)^{n+r} U_r^2 U_{n-r} s(n,k) = t(n,k+2).$$

This completes the proof. □

As a consequence of this theorem, we can see that the matrix $H(n)$ generate the $s(n,k)$. Since the elements of $H(n)$ are integers, the quotient $s(n,k)$ are integers.

Also from Theorem 1 in [2], we have the following result for the combinatorial representation of $s(n,k)$.

Corollary 1.

$$s(n,k) = \sum_{(l_1,l_2,l_3)} \binom{l_1+l_2+l_3}{l_1,l_2,l_3} (-1)^{(n+r-1)l_3} A_{n-1}^{l_1} B_n^{l_2} U_r^{2l_3} U_{n-r}^{l_3},$$

where the summation is over nonnegative integers satisfying $l_1 + 2l_2 + 3l_3 = k-2$.

As another main result, we have the following theorem.

Theorem 2. *For $n,k \geq 1$*

$$(G(n,k))_{3,1} = s(n,k) =$$
$$= \frac{(-q)^{n-r}U_r^k U_{n(k-1)} + (-1)^k(-q)^{r(k-1)}U_n U_{n-r}^k + U_r^{k-1}U_{kn}U_{n-r}}{U_n^3}.$$

Proof. Since the eigenvalues of $H(n)$ are distinct, $H(n)$ is diagonalizable as

$$V^{-1}H(n)V = D,$$

where

$$V = \begin{bmatrix} U_r^2\alpha^{2n} & U_r^2\beta^{2n} & q^{2r}U_{n-r}^2 \\ U_r\alpha^n & U_r\beta^n & -(-q)^r U_{n-r} \\ 1 & 1 & 1 \end{bmatrix},$$

and $D = diag(U_r\alpha^n,\ U_r\beta^n,\ -(-q)^r U_{n-r}\ )$. Therefore, we obtain $V^{-1}H(n)^kV = D^k$. By Theorem 1, we write $V^{-1}G(n,k)V = D^k$. Then we have the following linear equation system:

$$g_{i1}U_r^2\alpha^{2n} + g_{i2}U_r\alpha^n + g_{i3} = U^{k+(3-i)}\alpha^{kn+(3-i)n}$$
$$g_{i1}U_r^2\beta^{2n} + g_{i2}U_r\beta^n + g_{i3} = U^{k+(3-i)}\beta^{kn+(3-i)n}$$
$$g_{i1}q^{2r}U_{n-r}^2 - g_{i2}(-q)^rU_{n-r} + g_{i3} = (-1)^{(r-1)(k+3-i)}q^{kr+(3-i)r}U_{n-r}^{k+(3-i)}$$

Using the identities

$$U_{n-r}U_{n+r} - U_n^2 = -(-q)^{n-r}U_r^2$$

and

$$q^rU_{n-r} + (-1)^rU_rV_n = (-1)^rU_{n+r},$$

the solution of the above linear equation system gives the claimed result. □

By considering definition of $s(n,k)$, we have the following consequence of Theorem 2.

Corollary 2. *Let n,k and r arbitrary integers. Then*

$$U_{n-r}U_{kn} = U_nU_{kn-r} - (-q)^{n-r}U_rU_{n(k-1)}.$$

The next result presents a similar expression by considering generalized Lucas sequence $\{V_n\}$.

Theorem 3. *For all integers n,k,r,*

$$U_{n-r}V_{kn} = U_nV_{kn-r} - (-q)^{n-r}U_rV_{n(k-1)}.$$

Proof. Using Binet formulas of the sequence $\{U_n\}$ and $\{V_n\}$, we have

$$U_nV_{kn-r} - (-q)^{n-r}U_rV_{n(k-1)} =$$
$$= (\alpha^{kn+n-r} - \beta^{kn+n-r} + \alpha^n\beta^{kn-r} - \alpha^{kn-r}\beta^n - (-q)^{n-r}\alpha^{kn-n+r}$$
$$+ (-q)^{n-r}\beta^{kn-n+r} - (-q)^{n-r}\alpha^r\beta^{kn-n} + (-q)^{n-r}\alpha^{kn-n}\beta^r)/(\alpha-\beta)$$

$$
\begin{aligned}
&= \left(\alpha^{kn+n-r} - \beta^{kn+n-r} - (-q)^{n-r}\alpha^{kn-n+r} + (-q)^{n-r}\beta^{kn-n+r}\right)/(\alpha-\beta) \\
&= (\alpha^{n-r} - \beta^{n-r})(\alpha^{kn} + \beta^{kn})/(\alpha-\beta) \\
&= U_{n-r}V_{kn}.
\end{aligned}
$$

The proof is complete. □

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